

Hypothesis Testing in Functional Data Analysis: a Non-parametric Approach

*Verifica delle ipotesi per l'analisi di dati funzionali: un
approccio non parametrico*

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Abstract We present the Interval Testing Procedure (ITP), a non parametric procedure based on permutation tests that enables inference for functional data. The procedure is based on an interval-wise control of the Family Wise Error Rate (FWER), which allows selecting the basis components which present a statistically significant test result. We present some theoretical results on the power and error control of the ITP. Then, by means of a simulation study, we empirically compare the power and the FWER of the ITP with the ones of the Benjamini-Hochberg procedure.

Abstract Presentiamo l'Interval Testing Procedure (ITP), una procedura non parametrica basata su test di permutazione che consente l'inferenza per dati funzionali. La procedura si basa su un controllo intervallare del Family Wise Error Rate (FWER), che permette di selezionare le componenti di base per le quali il risultato del test è statisticamente significativo. Vengono riportati alcuni risultati teorici sul controllo dell'errore e sulla potenza dell'ITP. Infine, attraverso uno studio di simulazione, vengono confrontati la potenza e il FWER dell'ITP con quelli della procedura di Benjamini-Hochberg.

Key words: Functional Data Analysis, Permutation Test, FWER, Multiplicity Correction

1 Introduction

Functional Data Analysis addresses the problem of analyzing a set of curves, which has become a more and more important area of scientific investigation in the recent years [2, 8, 9]. In particular, the problem of making inference for functional data has been approached in different ways. When inference has to be performed on

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curves, usually the simple rejection of the null hypothesis is no longer a sufficient result from an application point of view. Indeed, if we know that for example the means of two functional populations are significantly different, we would also like to understand in which part of the domain the differences are observed. It may be throughout all the domain, or only in a particular zone; it can be expressed on all frequencies, or can be an effect of only some particular bands of frequencies. In general, selecting the characteristics of the curves that lead to the rejection, is often as important as the rejection of the null hypothesis itself.

For this purpose, we present the Interval Testing Procedure (ITP), a newly proposed inferential methodology enabling to select, in case of rejection, the curve characteristics that lead to the rejection of the null hypothesis. The methodology is based on the expansion of the data on a functional basis, and can be applied to any situation which a scalar counterpart exists (i.e., inference for the mean of one or several populations, inference for the variance of two or several populations, etc.). The procedure is described in detail in [6], and an R package implementing it in the case of testing the mean of one or two populations is available on CRAN [7].

2 Methodology

The ITP is a non-parametric inferential procedure that is articulated as follows:

- A functional basis (even high-dimensional, if necessary) is selected (e.g., Fourier, B-splines) and data are represented by means of the coefficients of the basis expansion.
- The significance of each basis component is tested with a suitable univariate permutation test on the associated coefficients. All tests are performed jointly, i.e., the permutations are the same on each component.
- The results of the joint univariate permutation tests are combined to build multivariate permutation tests (by means of the Non Parametric Combination test presented in [5]) on all intervals of basis components.
- An adjusted p -value for each basis component is calculated by taking the maximum p -value on each univariate and multivariate test pertaining to that component.

The adjusted p -values obtained with this methodology provide an interval-wise control of the Family Wise Error Rate, stating that the probability of wrongly rejecting any closed interval of basis components associated to all true null hypotheses is controlled. Hence, the ITP provides, as a result, the selection of the statistically significant basis components.

Note that the procedure may be applied to build a test in a wide range of different situations, provided that we are able to choose a suitable univariate permutation test for each component. Thus, by changing the permutations and the test statistic of the univariate permutation tests, we can build an ITP for the mean of a single population, for the mean difference between two, or more than two populations, for

the inequality between variances, for testing the significance of a linear model, and for many other situations.

As a final remark, the freedom in the choice of the functional basis to be used at the first stage of the procedure enables to build very different types of tests, depending on the structure of the problem at hand. For example, if we want to select the significantly different regions between two families of curves, a local basis as the B-splines may be used. In this case, significantly different basis components correspond to significantly different regions of the domain, and the interval-wise control of the FWER allows to state that, for any interval in which there is no difference between the functional populations, the probability of rejecting it (or part of it) as significant is always controlled. If, on the other hand, we aim at detecting differences in the frequency domain, we shall use a Fourier basis in order to catch amplitude and phase differences at the explored frequencies. In this case, the control is for any band of the frequency domain.

3 Results

In addition to showing that the ITP is provided with an interval-wise control of the FWER, in [6] we prove that: (i) the global level of the ITP is bounded above by the global level of the Global Testing Procedure (which however provides only a weak control of the Family Wise Error Rate and does not provide any guide to the interpretation of the test result) and from below by the global level of the Closed Testing Procedure [4] (which provides a strong control of the Family Wise Error Rate but it is computationally unfeasible in the functional framework). The same result holds for the global power of the procedure; (ii) marginally for each component, the level of the ITP is bounded above by the level of the global test and from below by the level of the Closed Testing Procedure. The same result holds for the marginal power.

Moreover, a simulation study on the comparison between the ITP and two other multiple testing approaches (the Bonferroni-Holm [3] and the Benjamini-Hochberg [1] procedures) is presented in [6]. Simulations empirically show that the Bonferroni-Holm procedure leads to an extremely conservative result if the number of basis components increases, making the correction useless from a practical perspective. On the other hand, the ITP seems to outperform the Benjamini-Hochberg procedure on all false hypotheses not occurring at the boundaries between “true” and “false” regions. This latter finding could suggest the use of the ITP when the “false” components are expected to be mostly grouped in intervals, as in the functional case, while the use of the Benjamini-Hochberg procedure when the latter ones are expected to be mostly sparse and isolated.

The ITP is applied to two different case studies: inference for the mean function of the daily temperature profiles in Milan (Italy) using a Fourier basis expansion based on the amplitude-phase decomposition; and inference for the difference between vascular geometry and hemodynamic features of the Internal Carotid Artery

of two groups of subjects with a different severity of aneurysm pathology, using a B-spline basis expansion.

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